

On Optimal Fee Structure

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Investors look at hedge funds as an opportunity to invest in uncorrelated with major asset classes, conservatively managed vehicles. Apart from redeeming from the fund, investors have no means to ensure that the manager "sails" in a conservative and prudent way. We argue here that investor's control of the fund money management can be effectively achieved by adjusting the pay-off structure, namely ratio of fixed and variable fees.

With a good degree of accuracy, terminal pay-off for a hedge fund manager can be modeled as a fixed (management) fee, variable (incentive) fee subject to inter-temporal down-and-out barrier conditions on the largest possible running loss and maximal tolerable realized volatility. In this case we find the optimal relation between fixed and variable fees such that the optimal parameters of the pay-off maximize the investors' utility function as defined by the expected terminal value of the investment conditional on not exceeding a particular volatility threshold, or by the Sharpe ratio.

Fee Structure as Investor's Control of Manager's Investment Policy

The hedge fund manager pay-off is a price that investors pay to a manager for the asset management service. The fund manager complies with stated investment policy. The investment policy specifies the types of financial instruments allowed for investment and the type of the arbitrage strategy, but leaves to the manager to decide on the allocation policy, i.e. the share of fund assets to be invested into risky instruments at any particular moment of time.

The investor's objective is to maximize the expected utility function, which can be either the total return on investment subject to a certain volatility constraint, Sharpe ratio or any other appropriate utility function. The objective of fund manager is to maximize his expected terminal pay-off. Fund managers are in the "driving sit" and notoriously secretive and independent. Here come the question: *How can investor influence the manager to achieve their desirable risk/return profile?*

The problem is studied in this paper. We investigate how investor can optimally choose the structure of the management fees so that he achieves his target investment profile¹.

¹The approach is based on Fusion Money Management Model developed in *Fusion Money Management Model (F3M): Quantitative Approach to Capital Allocation in Arbitrage Trading*, Fusion Asset Management Technical Document 01-04, Aug 2004.

Hedge fund manager pay-off is usually defined by the value of the fund $F_T = F(T)$ at the end of investment term (the terminal value) and a set of barrier conditions that must hold at any time during the investment term (inter-temporal conditions). This pay-off consists of fixed and variable components, $P(T) = P_1(T) + P_2(T)$. The fixed amount is a management fee P_1 , it equals to a certain share of the fund notional. The variable part is called an incentive fee, it equals to a certain share of the excess return over the target fund value level. The management fee has a form of binary option pay-off with strike $S_b = s_b F(0)$ and notional $N_b = n_b F(0)$, the incentive fee P_2 , coincides with the pay-off of vanilla call option with a strike S_v ($F(0)$ plus fixed fee) and notional $N_v = n_v F(0)$.

$$P(F) = N_b \theta(F - S_b) + N_v \theta(F - S_v) (F - S_v). \quad (1)$$

Fund manager receives the pay-off only if a set of inter-temporal conditions on the state of the fund is satisfied during the investment term. The usual conditions are:

1. Down-and-out barrier on the fund value (so called "red card event"), meaning that if at any time during investment term between time 0 and time T the fund value falls below the lower bound $B_F = b_F F(0)$ investors withdraw money and the manager closes the shop. The parameter B_F usually equals to the binary strike S_b in management fee, i.e. while investors keep money with the fund they pay the management fee.
2. Up-and-out barrier condition on the fund realized volatility². If the realized variance exceeds the upper bound B the pay-off is zero: the fund is perceived as too risky for its class and investors redeemed from the fund.

Therefore in this simple but realistic case the manager pay-off is completely defined by a set of five parameters $n_b, n_v, B_F = s_b, s_v, B_F, B$. In practice, only two parameters, n_b and n_v , are negotiable.

The parties can agree on the total expected level of reward using the average across the industry fee levels, but the same level of reward can be achieved through various combinations of fee's magnitudes n_b and n_v . **The relative value of fixed fee to the variable fee is the most effective fee structure parameter to determine fund investment policy and to allow investor to control the manager's money management.**

To demonstrate this, suppose that the fixed fee is relatively high comparing to the variable fee. For the fund manager the reward from receiving the variable part of pay-off on top of fixed fee may not compensate for the risk to break the barriers and lose it all. The manager is likely to choose more conservative money management (lower capital allocation). The main danger for investor is that the lack of incentive may decrease the total investor return.

Now let us consider the situation when the contribution of fixed fee is small comparing to that of the variable fee. Such structure will make the fund manager more incentivized to "swing the bat", prompting to load exposure and go for the variable fee. The risk of breaking the barriers and losing the relatively small fixed fee may become too low to prevent the manager from taking high risk to "blow up", allocating more to the risky assets³. The investor return may be higher in this case, however the risks to exceed the loss or variance barriers are higher too.

These intuitive examples show that the fee structure determines the allocation policy of the fund manager. The same general level of compensation but different split between fixed and variable fees lead to different allocation policies and corresponding different risk-return profiles of the investment. In general, **investors with high risk tolerance will go for lower proportion of fixed fee and higher proportion of performance fee. At the same time, investors concerned with capital preservation, low perceived risk and conservative investment should go for a larger proportion of fixed fees.**

²Below we use variance $I(t)$ instead: the final condition for volatility and variance are equivalent but variance can be treated analytically.

³This is typical situation of a trader in investment bank: fixed fee (salary) is order of magnitude less than the potential variable fee (bonus).

In the next section we briefly describe the procedure for finding the optimal allocation policy for a given fee structure. In the last section we perform Monte Carlo simulations to find the expected terminal fund value under optimal allocation policy and calculate investors utility function for various combinations of key parameters resulting in the same fund manager expected pay-off.

Mathematics of Optimal Allocation

We can formulate the problem of optimal fee structure as following:

find the values of n_b , n_v such that for a given level of fund manager expected pay-off deliver the maximum to the investor utility function.

To this end we assume that the manager implements optimal allocation for arbitrage trading strategy. The general model of the fund value is given by the following stochastic process:

$$dF(t) = a_S dS + a_D dD = \left(w(t) \frac{dS}{S} + (1 - w(t)) \frac{dD}{D} \right) F(t)$$

where $w(t)$ is allocation to the strategy (Control Function), $D(t)$ is a riskless cash deposit and $S(t)$ is a risky log-normal asset (P&L of the pure trading strategy)

$$\frac{dD(t)}{D(t)} = r dt, \quad \frac{dS(t)}{S(t)} = \mu dt + \sigma dB(t)$$

In order to control the variance of the fund an additional state variable, the realized variance $I(t)$ is introduced in the usual manner: by definition

$$dI(t) = E \left[\left(\frac{dF}{F} \right)^2 | F(t) \right] = \sigma^2 w^2(t) dt$$

The optimized Value Function depends on all state variables $F(t)$, $I(t)$ and the Control Function $w(t)$. Substituting the expressions for $dS(t)$ and $dD(t)$ into the process for the fund we get the following state equation:

$$\frac{dF(t)}{F} = \left(w(t)\mu + (1 - w(t)r) \right) dt + w(t)\sigma dB(t) ,$$

where $w(t, F, I)$ is an allocation function implemented by the manager. We assume that the allocation is chosen optimally by the manager, i.e. so as to maximize the fund manager expected terminal pay-off.

Value Function, that satisfies the Hamilton-Jacobi-Bellman (HJB) nonlinear PDE

$$V_t + rFV_F + \max_{w \in [0,1]} \left\{ w(\mu - r)FV_F + \frac{1}{2}w^2\sigma^2F^2(V_{FF} + 2V_I) \right\} = 0 , \quad (2)$$

with the budget constraint is then⁴

$$0 \leq w \leq 1 . \quad (3)$$

The intertemporal conditions provide the boundary conditions for the PDE. The barrier condition on the fund value can be written as:

$$V(S_b, I(t), t) = 0 . \quad (4)$$

Since by definition the realized variance is monotonously increasing function of time, the variance barrier condition can be written in the terminal form. Together with the terminal pay-off condition the two can be combined as

$$V(F(T), I(T), T) = P(F(T))\vartheta(B - I(T)) . \quad (5)$$

⁴We assume that the fund manager can not borrow funds besides permitted leverage (margin trading) already included in the definition of the trading strategy.

The barrier-terminal problem for the HJB equation is solved point-wise. On each iteration the solution of the barrier-terminal problem for linearized PDE (without max-operator) is found numerically for a particular value of allocation a . After that the point $w^*(a)$ where $\phi(w)$ reaches maximum is found. By bisection method the solution of $a = w(a)$ is found, giving the optimal allocation for a given spot, realized variance and time. As a result we find the optimal allocation on a time-space grid. The allocations for the points between nodes are obtained by the linear interpolation. The detailed description of the procedure is given in [].

Optimal Fee Structure

To find the optimal fee structure we perform Monte Carlo simulations for fund stochastic process corresponding to the optimal allocation process resulting from various combinations of magnitudes of fixed and variable pay-off components subject to the same (locked) expected manager pay-off.

The two arbitrary pairs $S_1 \equiv \{n_{b,1}, n_{v,1}\}$ and $S_2 \equiv \{n_{b,2}, n_{v,2}\}$ of management fee rate n_b and incentive fee rate n_v in general lead to different values of expected fund manager pay-off. In order to be able to compare the two pairs representing the different fee structures we must put the pairs on the surface of constant expected manager pay-off, the so-called iso-VF surface.

The two limit points on the surface are purely management fee pay-off and purely incentive fee pay-off. Incentive-free fund manager must and will do exactly nothing to guarantee himself the maximum possible pay-off, which equals to the management fee. Any non zero allocation threatens with non-zero chances to break the barriers without giving any sensible benefits. The zero management fee ($n_b = 0, n_v \neq 0$) is another case when the pay-off can be easily estimated. If the volatility of the risky asset is not very high then the region where the optimal allocation differs from the maximum level w_{max} (defined by the variance barrier) is very narrow. The expected pay-off is close to the expected value of a portfolio consisting of w_{max} of risky asset and $1 - w_{max}$ of cash deposit. If the variance barrier is not very restrictive then $w = w_{max} = 1$ and the expected pay-off equals approximately to $n_v(\mu - \frac{\sigma^2}{2})$

The straight line running through the two limit pairs $(\lambda, 0)$ and $(0, \frac{\lambda}{\mu - \frac{\sigma^2}{2}})$ is quite a good approximation of iso-VF surface.

The residual deviation of iso-VF surface from the straight line is fitted with quadratic function with very good precision.

To eliminate the residual deviation we use re-scaling procedure. The simultaneous rescaling of the fee rates adjusts the level of pay-off but preserves the allocation function. It follows from the definition of Value Function and the linearity of the max-operator.

Indeed, by definition the expected pay-off at given time and state is the Value Function. The Value Function is a solution of barrier-terminal problem for HJB equation and has the following uniformity property **if $V(n_b, n_v)$ is a value function for the pay-off defined by the fixed fee magnitude n_b and variable fee magnitude n_v then the function $aV(n_b, n_v)$ is a value function for the pay-off with fees magnitudes an_b, an_v** . The proof follows from the uniformity of max-operator and uniformity of the terminal condition:

$$\arg \max_x [af(x)] = \arg \max_x f(x) \quad \text{and} \quad P(an_b, an_v) = aP(n_b, n_v) .$$

Now, to complete the procedure for building fee structure we must take into account one more condition. Strictly speaking, the pay-off parameters are not completely independent. The incentive fee usually starts paying only after the increase in the the fund value has compensated for the management fee. It means that the incentive fee strike equals to $1 + n_b$. If the management fee rate changes then the incentive fee strike follows. Thus, the fee structure is defined by the pair of fee rates and the related strike.

Rescaling adjusts the pay-of level and preserves the allocation function, but it does not preserves the relation between the management fee and incentive strike. Nevertheless, the changes in the fee rates to

the pairs on the straight line approximation from the rescaling are small in relative terms, and we can consider the fee structure to stay unchanged under rescaling.

We performed simulations for three different values of Sharpe ratio of the risky asset. The Sharpe ratio of the first asset is 0.5 ($\mu = 15\%$, $\sigma = 30\%$). The asset used in the second simulation run is 1.0 ($\mu = 30\%$, $\sigma = 30\%$). In the last run we used the asset with Sharpe ratio of 1.5, ($\mu = 45\%$, $\sigma = 30\%$). For each run we used the following parameters:

Risk free rate: $r = 3\%$, expected pay-off level: $\lambda = 5\%$. Pay-off: fund barrier $B_F = s_b = 0.9$, variance barrier at 75% of σ^2 . Numerics parameters: 20 fee steps, 12 time steps, 20 variance steps, 100 spot steps, 10^6 simulation trials.

Asset: drift = 15%, volatility = 30%

Ratio	\bar{R}	σ_R	$\bar{\sigma}_R$	Sharpe
1.85%	4.70%	17.18%	3.41%	0.688
2.27%	4.69%	17.21%	3.34%	0.703
2.71%	4.67%	17.23%	3.27%	0.713
3.17%	4.63%	17.24%	3.24%	0.714
3.66%	4.62%	17.29%	3.22%	0.718
4.17%	4.56%	17.28%	3.22%	0.708
4.70%	4.56%	17.33%	3.21%	0.709
5.26%	4.53%	17.35%	3.24%	0.699
5.86%	4.49%	17.40%	3.24%	0.692
6.48%	4.48%	17.43%	3.27%	0.685
7.14%	4.43%	17.46%	3.29%	0.674
7.84%	4.42%	17.53%	3.31%	0.668
8.59%	4.40%	17.56%	3.35%	0.657
9.38%	4.38%	17.62%	3.37%	0.649
10.22%	4.36%	17.66%	3.41%	0.640
11.11%	4.33%	17.72%	3.44%	0.631
12.07%	4.30%	17.77%	3.46%	0.621
13.10%	4.30%	17.84%	3.50%	0.614
14.20%	4.26%	17.90%	3.53%	0.603
15.38%	4.25%	17.95%	3.57%	0.596
16.67%	4.24%	18.02%	3.61%	0.587

Asset: drift = 30%, volatility = 30%

Ratio	\bar{R}	σ_R	σ_R^-	Sharpe
1.85%	15.80%	20.22%	4.02%	1.964
2.27%	15.78%	20.33%	3.96%	1.990
2.71%	15.73%	20.45%	3.88%	2.026
3.17%	15.67%	20.56%	3.83%	2.044
3.66%	15.63%	20.67%	3.77%	2.073
4.17%	15.57%	20.77%	3.74%	2.081
4.70%	15.54%	20.88%	3.72%	2.089
5.26%	15.51%	20.99%	3.71%	2.093
5.86%	15.45%	21.09%	3.70%	2.088
6.48%	15.44%	21.18%	3.71%	2.081
7.14%	15.37%	21.30%	3.70%	2.078
7.84%	15.33%	21.38%	3.70%	2.072
8.59%	15.31%	21.51%	3.71%	2.065
9.38%	15.24%	21.60%	3.73%	2.042
10.22%	15.25%	21.71%	3.73%	2.045
11.11%	15.19%	21.83%	3.75%	2.024
12.07%	15.13%	21.91%	3.78%	2.003
13.10%	15.14%	22.05%	3.78%	2.001
14.20%	15.06%	22.12%	3.80%	1.980
15.38%	15.08%	22.27%	3.82%	1.971
16.67%	15.01%	22.36%	3.85%	1.951

Asset: drift = 45%, volatility = 30%

Ratio	\bar{R}	σ_R	σ_R^-	Sharpe
1.85%	28.77%	21.69%	4.56%	3.156
2.46%	28.71%	21.88%	4.49%	3.196
3.11%	28.69%	22.14%	4.37%	3.280
3.80%	28.59%	22.38%	4.30%	3.324
4.55%	28.56%	22.56%	4.25%	3.363
5.35%	28.52%	22.83%	4.19%	3.404
6.21%	28.44%	23.01%	4.19%	3.398
7.14%	28.43%	23.24%	4.15%	3.423
8.16%	28.35%	23.45%	4.14%	3.424
9.26%	28.31%	23.65%	4.14%	3.419
10.47%	28.27%	23.90%	4.13%	3.419
11.79%	28.21%	24.07%	4.14%	3.407
13.25%	28.20%	24.32%	4.15%	3.396
14.86%	28.13%	24.48%	4.17%	3.375

One can see that under locked manager pay-off of 5% the expected fund value decreases with the fee ratio and the volatility increases. The expected investor return Π also decreases with the growth of fee ratio, but the standard deviation of the negative investor returns reaches minimum inside the fee ratio interval.

The position of maximum of Sharpe ratio of investor return depends on the Sharpe ratio of the asset. The higher the asset's ratio the larger the value of fee ratio where investor's ratio reaches maximum.

For the particular assets that was evaluated we found that the optimal fee ratio for the asset with Sharpe ratio of 0.5 the optimal fee ratio is about 3.7%. For the asset with Sharpe ratio that equals 1.0 the optimal fee ratio is 5.3%. The asset with Sharpe ratio of 1.5 results in the maximum at 8.2%.

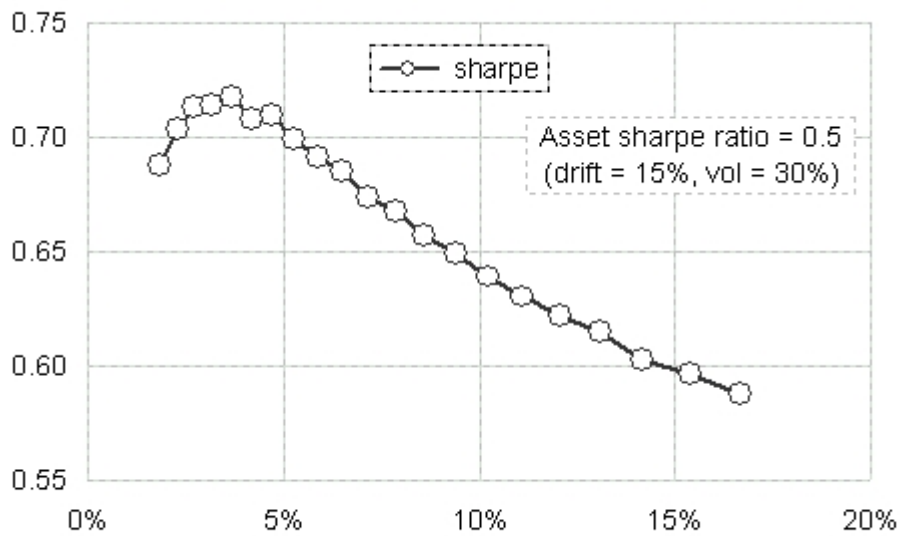


Figure 1: Sharpe ratio as a function of fee ratio

The general level of the values that takes the Sharpe ratio of investor returns grows with the Sharpe ratio of the asset. Since the downside is bounded (the fund can go below the lower barrier) the fund distribution is strictly non-normal and the value of Sharp ratio can not be compared directly with that of a normally distributed return.

Summary

We showed that the more the relative weight of the incentive fee in manager's pay-off the higher the expected investor return. However, the investor is better off if a certain balance between fixed management fee and variable incentive fee is chosen. The value of the optimal fee ratio that maximizes the investor utility measured by Sharpe-like ratio of expected return and standard deviation of negative investor returns depends monotonously grows with the Sharpe ratio of the asset.

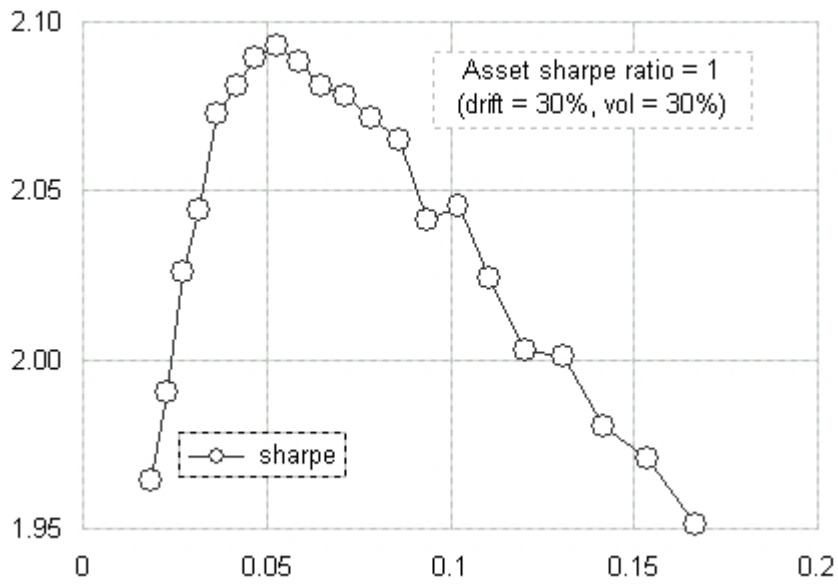


Figure 2: Sharpe ratio as a function of fee ratio

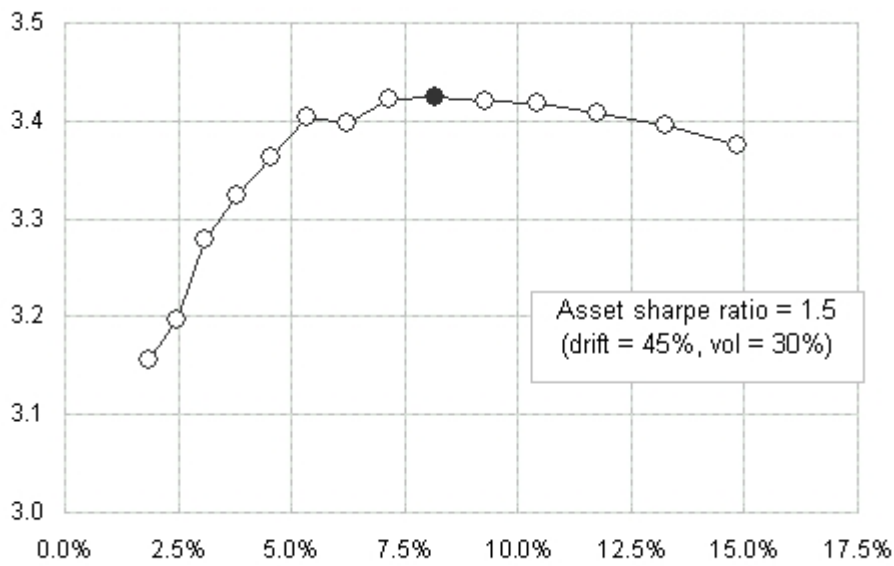


Figure 3: Sharpe ratio as a function of fee ratio